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Flow past periodic arrays of spheres at low Reynolds number

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We calculate the force on a periodic array of spheres in a viscous flow at small Reynolds number and for small volume fraction. This generalizes the known results for the force on a periodic array due to Stokes flow (zero Reynolds number) and the Oseen correction to the Stokes formula for the force on a single sphere (zero volume fraction). We use a generalization of Hasimoto's approach that is based on an analysis of periodic Green's functions. We compare our results to the phenomenological ones of Kaneda for viscous flow past a random array of spheres.

1. Introduction

Inertial effects on particle motion in low-Reynolds-number flow are of interest in many applications but their theoretical analysis is rather complicated, even for a single particle, as shown by Lovalenti & Brady (1993). In this paper we calculate inertial corrections to the hydrodynamic force on a fixed periodic array of spheres in steady, viscous and incompressible flow. All spheres have the same radius a and their centres are placed on a cubic lattice of span L. The volume fraction

$$c = 4\pi a^3/3L^3 \tag{1.1}$$

occupied by the spheres is assumed to be small as is the Reynolds number $Re = U_0 a/\nu$, which is based on the average flow rate $\boldsymbol{U}_0$ of the fluid past the spheres, with $U_0 = |\boldsymbol{U}_0|$ and ν the kinematic viscosity.

When both the volume fraction and the Reynolds number are infinitesimal, we have viscous flow past a single sphere with no inertial effects. The force is then given by the Stokes formula (Batchelor 1967)

$$\boldsymbol{F} = 6\pi\mu a\boldsymbol{U}_0. \tag{1.2}$$

When the Reynolds number is small, inertial effects appear with the Oseen correction

$$\boldsymbol{F} = 6\pi\mu a\boldsymbol{U}_0(1 + \tfrac{3}{8}Re). \tag{1.3}$$

This was analysed in detail by Proudman & Pearson (1957) using matched asymptotic expansions.

When the Reynolds number is zero and the volume fraction c occupied by the periodic array of spheres is small, the force on the array was studied by Hasimoto (1959) and his result for the simple cubic lattice is

$$\boldsymbol{F} = 6\pi\mu a \boldsymbol{U}_0(1 + 1.7601c^{1/3}). \tag{1.4}$$

This is one of several results concerning dilute suspension of small spheres in a viscous fluid (Batchelor 1972; Brinkman 1947; Childress 1972; Saffman 1973; Zick & Homsy 1982).

In this paper we calculate from first principles the small inertial corrections to Hasimoto's formula (1.4) when the Reynolds number is small but not zero, and the volume fraction c is also small but not zero. Kaneda (1986) studied this problem for a random array of fixed spheres. He started with Brinkman's equation (Brinkman 1947) that describes flow in a fixed random suspension of spheres, an effective equation, and added to it inertial effects just as in the Oseen calculation. The Brinkman equation is reasonably well understood as an effective equation (Hinch 1977; Rubinstein 1986), but a mathematical justification for it, especially with inertial effects, is hard and unavailable. Kaneda (1986) obtained the following formula for the drag:

$$\boldsymbol{F} = 6\pi\mu a \boldsymbol{U}_0[1 + Re\hat{F}(S)], \tag{1.5}$$

where

$$\hat{F}(S) = \frac{3}{8}\left[(2S+1)(4S+1)^{1/2} - 4S^2 \ln\frac{(4S+1)^{1/2}+1}{(4S+1)^{1/2}-1}\right], \tag{1.6}$$

and $S = 9c/(2Re^2)$. This result is consistent with Oseen's formula (1.3) when $c = 0$ and with Brinkman's formula

$$\boldsymbol{F} = 6\pi\mu a \boldsymbol{U}_0\left(1 + \frac{3c^{1/2}}{\sqrt{2}}\right) \tag{1.7}$$

when $Re = 0$. Note the characteristic difference between fixed periodic and random arrays where the force depends on $c^{1/3}$ in the periodic case and on $c^{1/2}$ in the random case, for c small.

Inertial effects and interacting sphere effects are not additive, even when the Reynolds number and volume fraction are small, because the equations are nonlinear. This is clearly seen in Kaneda's result (1.5) although it is not discussed in detail in Kaneda (1986). In figure 1 we plot the relative difference between the additive effects of inertia and particle flow interaction, and Kaneda's formula (1.5). This relative difference is defined as

$$E(Re, c) = \frac{\left(\frac{3}{8}Re + \frac{9}{2}c\right) - Re\hat{F}(S)}{Re\hat{F}(S)}, \tag{1.8}$$

which can also be written in the form

$$E(Re, c) = \frac{\left(\frac{3}{8} + S^{1/2}\right)}{\hat{F}(S)} - 1.$$

This implies that $E(Re, c)$ is a constant along $S =$ constant, i.e. $Re/c^{1/2} =$ constant in the $(Re, c^{1/2})$-plane. This is not very clear from the black and white figure 1, but can be seen easily from a coloured picture in which the colour shows the height of $E(Re, c)$. We note from the figure that E is positive, so that fluid–particle interaction reduces drag, and that it can be as large as 40%. A similar 'screening' effect is observed in the calculation of heat transfer in a dilute fixed bed of spheres at a fixed temperature (Acrivos, Hinch & Jeffrey 1980).

We analyse here the periodic version of Kaneda's problem starting from the

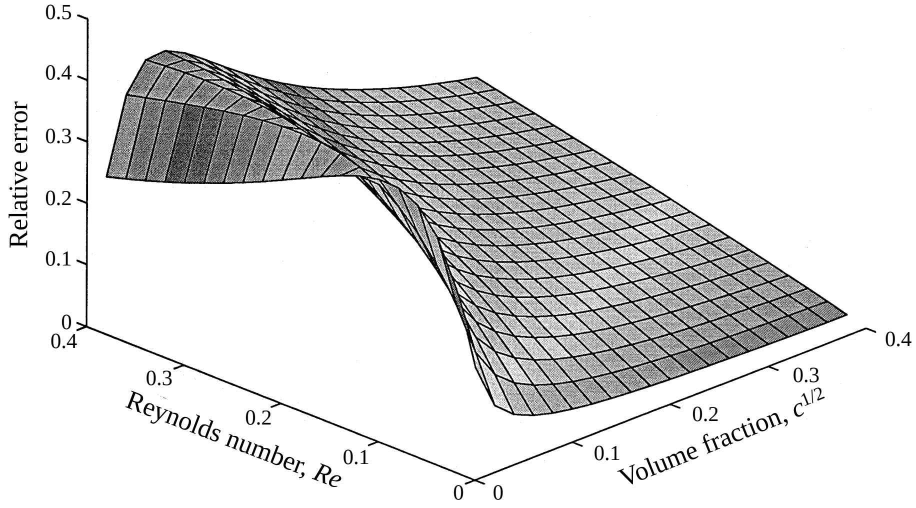

FIGURE 1. Surface plot of the relative difference between the additive effects of inertia and particle flow interaction, and Kaneda's results (1.5).

steady Navier–Stokes equations and using matched asymptotic expansions (Lagerstrom 1988), combined with a generalized form of Hasimoto's method of periodic Green's functions. For simplicity we consider only the case of a simple cubic lattice of spheres with centres at lattice points $\boldsymbol{x}_{\boldsymbol{n}} = L(n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3)$ for $\boldsymbol{n} = (n_1, n_2, n_3)$ integers. Our main result is the following formula for the force on the array:

$$\boldsymbol{F} = 6\pi\mu a\boldsymbol{U}_0 \left(\boldsymbol{I} + \tfrac{3}{8}Re\boldsymbol{I} + \tfrac{3}{2}Re[C(\theta)\boldsymbol{I} + \boldsymbol{M}(\theta)]\right) + ... \tag{1.9}$$

where $C(\theta)$ and $\boldsymbol{M}(\theta) = (C_{ij}(\theta))_{3\times 3}$ are given by (6.23) to (6.25) and $\boldsymbol{I}$ is the identity matrix; $\theta = LRe/a$ and in terms of volume fraction c, $\theta = Re/(3c/4\pi)^{1/3}$. When inertia is negligible and c is small we show that (1.9) reduces to Hasimoto's formula (1.4). In the opposite limit, where inertial effects dominate particle interaction, (1.9) reduces to the Oseen formula (1.3). A table of values for (1.9) is provided in §8.

In the intermediate regime where both inertial and particle interaction effects are important, (1.9) is not the simple addition of the two effects. This is shown in figure 2 which is qualitatively similar to figure 1. We plot the relative difference between the additive Oseen–Hasimoto effects and our result (1.9) as a function of particle radius a and Re, defined similarly to (1.8), which can be written as

$$E(Re, c) = \frac{1.1735(4\pi/3)^{1/3}}{\theta[C(\theta) + C_{11}(\theta)]} - 1.$$

The error depends on Re and c through the parameter θ. From figure 2, we see clearly the drop in the relative drag correction, the screening effect that is due to the fluid–particle interaction. The wiggles in the figure are due to numerical errors in calculating $C(\theta)$ and $C_{11}(\theta)$.

The paper is organized as follows. In the next section we formulate the problem. In §3 we review briefly Hasimoto's method. Sections 4 to 7 are devoted to the derivation of our results. In §8 we present some numerical results that illustrate how the force and the average velocity are related by the new formula (1.9). Some technical mathematical calculations are presented in the Appendices.

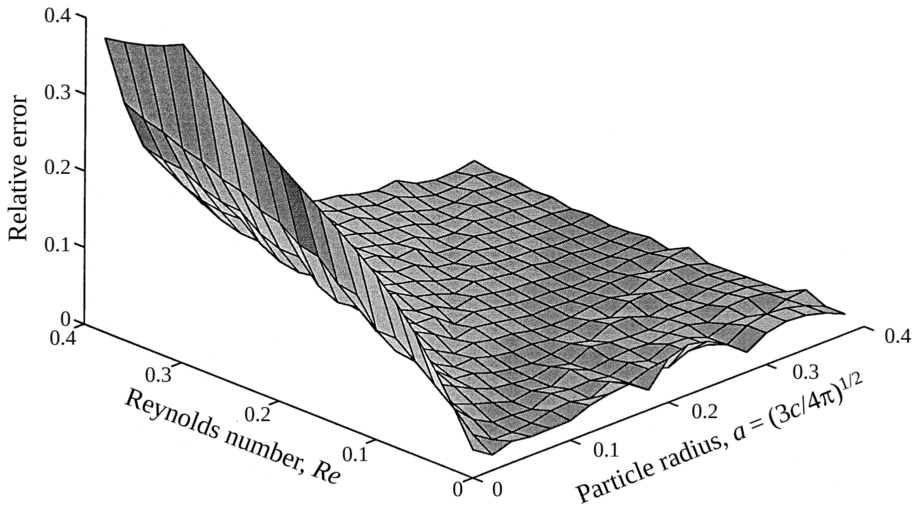

FIGURE 2. Surface plot of the relative difference between the additive Hasimoto-Oseen effects and our result (1.9).

2. Formulation of the problem

We consider a periodic array of identical rigid spherical particles of radius a in a Newtonian fluid of viscosity μ and density ρ, driven by an average pressure gradient. We wish to find the average fluid flow that results when a no-slip boundary condition is satisfied on the surface of the spherical particles. The flow satisfies the steady Navier–Stokes equations outside the particle array

$$\left.\begin{aligned} &\mu\nabla^2\boldsymbol{u} - \nabla p = \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \\ &\nabla\cdot\boldsymbol{u} = 0 \quad \text{for } |\,\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{n}}\,| > a, \quad \forall\boldsymbol{n}, \\ &\boldsymbol{u} = 0 \qquad\quad \text{for } |\,\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{n}}\,| = a, \quad \forall\boldsymbol{n}, \end{aligned}\right\} \tag{2.1}$$

with the requirements that $\boldsymbol{u}$ and ∇p be periodic. Here ∇^2 is the Laplace operator. Our goal is to calculate the inertial correction to Hasimoto's formula (1.4), which gives the relation between the average flow rate $\boldsymbol{U}_0$ and the drag force per particle $\boldsymbol{F}$:

$$\left.\begin{aligned} \boldsymbol{U}_0 &= |V|^{-1}\int_V \boldsymbol{u}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \\ \boldsymbol{F} &= \int_{|\boldsymbol{x}|=a} \boldsymbol{\Sigma}\cdot\boldsymbol{n}\,\mathrm{d}s. \end{aligned}\right\} \tag{2.2}$$

Here $V = [-L/2, L/2]^3 - \{|\boldsymbol{x}| < a\}$ is the cube of side L minus the sphere of radius a and $\boldsymbol{\Sigma}$ is the viscous stress tensor

$$\Sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

with

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

the rate of strain tensor. It is convenient to consider $\boldsymbol{F}$ as given and to attempt to determine $\boldsymbol{U}_0$. In the near-linear regime, where inertial effects are small, this relation is invertible.

Let us consider the flow as a perturbation of a uniform one due to the presence of the particle array. If the Renolds number $Re = U_0 a/\nu$, $\nu = \mu/\rho$, is not zero, then the flow is not correctly described by the Stokes equations (Batchelor 1972; Lovalenti & Brady 1993; Proudman & Pearson 1957) in the wake behind the particles, the Oseen region. The local inertia term is

$$\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\rho(\boldsymbol{U}_0\cdot\nabla)(\boldsymbol{U}_0-\boldsymbol{u}) + \rho((\boldsymbol{U}_0-\boldsymbol{u})\cdot\nabla)(\boldsymbol{U}_0-\boldsymbol{u}), \tag{2.3}$$

and when we have a dilute suspension, $\boldsymbol{U}_0 - \boldsymbol{u}$ may be approximated by a Stokeslet placed at a sphere centre. Then, the first of the two terms on the right-hand side of (2.3) behaves like $\rho U_0^2 a/r^2$, whereas the second behaves like $\rho U_0^2 a^2/r^3$. Compared with the viscous force, $\mu U_0 a/r^3$, the second term is always negligible if $Re \ll 1$. However, the ratio of the first term and the viscous force is

$$\frac{\rho U_0^2 a}{r^2}\Bigg/\frac{\mu U_0 a}{r^3} = \frac{r}{a}Re,$$

and is not small when $r \sim O(a/Re)$, which is called the Oseen distance. In our case, the distance between particles is L and so when $Re \sim O(a/L) = O(c^{1/3})$ the flow is not correctly described by the Stokes equations. We have to deal with the Navier–Stokes equations. We will use matched asymptotic expansions and Hasimoto's periodic Green's functions for this purpose. His work is based on the Stokes equations

$$\left.\begin{array}{ll} \mu\nabla^2\boldsymbol{U} - \nabla p = 0, & \\ \nabla\cdot\boldsymbol{u} = 0 & \text{for } |\,\boldsymbol{x}-\boldsymbol{x_n}\,| > a, \quad \forall\boldsymbol{n}, \\ \boldsymbol{u} = 0 & \text{for } |\,\boldsymbol{x}-\boldsymbol{x_n}\,| = a, \quad \forall\boldsymbol{n}, \end{array}\right\} \tag{2.4}$$

and is reviewed briefly in the next section.

In the rest of the paper we will deal with the dimensionless form of the Navier–Stokes equations

$$\left.\begin{array}{ll} \bar\nabla^2\bar{\boldsymbol{u}} - \bar\nabla p = Re(\bar{\boldsymbol{u}}\cdot\bar\nabla)\bar{\boldsymbol{u}}, & \\ \bar\nabla\cdot\bar{\boldsymbol{u}} = 0 & \text{for } |\,\bar{\boldsymbol{x}}-\bar{\boldsymbol{x}}_{\boldsymbol{n}}\,| > 1, \quad \forall\boldsymbol{n}, \\ \bar{\boldsymbol{u}} = 0 & \text{for } |\,\bar{\boldsymbol{x}}-\bar{\boldsymbol{x}}_{\boldsymbol{n}}\,| = 1, \quad \forall\boldsymbol{n}, \end{array}\right\} \tag{2.5}$$

where the dimensionless quantities are defined by

$$\bar{\boldsymbol{x}} = \frac{\boldsymbol{x}}{a}, \quad \bar{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U_0}, \quad \bar{p} = \frac{p}{\mu U_0/a}, \quad Re = \frac{\rho a U_0}{\mu},$$

and

$$\bar{\boldsymbol{x}}_{\boldsymbol{n}} = \frac{L}{a}(n_1, n_2, n_3).$$

The average flow velocity is now $\boldsymbol{e}_0 = \boldsymbol{U}_0/U_0$ which is a unit vector. We will omit the bars in what follows.

3. Brief review of Hasimoto's method

Hasimoto (1959) uses the periodic fundamental solutions of problem (2.4) defined by

$$\left.\begin{aligned} \mu\nabla^2 \boldsymbol{u} &= \nabla p + \boldsymbol{F}\sum_n \delta(\boldsymbol{x}-\boldsymbol{x}_n), \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned}\right\} \tag{3.1}$$

This may be used to construct a systematic expansion for the flow for small volume fraction (as in Sangani & Acrivos 1983 for the analogous diffusion problem). It is also the lowest-order term in this expansion and it is then called the point-force approximation, since the spherical inclusions are replaced by point forces (3.1). Let

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}_{\boldsymbol{k}} \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}, \tag{3.2}$$

$$-\nabla p = \sum_{\boldsymbol{k}} \hat{\boldsymbol{p}}_{\boldsymbol{k}} \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}, \tag{3.3}$$

where $\boldsymbol{k}$ has integer components. Then, from (3.1)

$$\begin{aligned} -4\pi^2\mu \mid \boldsymbol{k} \mid^2 \hat{\boldsymbol{u}}_{\boldsymbol{k}} &= -\hat{\boldsymbol{p}}_{\boldsymbol{k}} + \boldsymbol{F}, \\ \boldsymbol{k}\cdot\hat{\boldsymbol{u}}_{\boldsymbol{k}} &= 0, \\ \boldsymbol{k}\times\hat{\boldsymbol{p}}_{\boldsymbol{k}} &= 0. \end{aligned}$$

Therefore, $\boldsymbol{u}$ and $-\nabla p$ have the representations

$$\begin{aligned} -(\nabla p)_j &= F_j - (4\pi)^{-1} F_l \frac{\partial^2 S_1}{\partial x_l \partial x_j}, \\ u_j &= U_{0j} - (4\pi\mu)^{-1}\left(F_j S_1 - F_l \frac{\partial^2 S_2}{\partial x_l \partial x_j}\right), \end{aligned}$$

for $j = 1, 2, 3$. Here

$$\begin{aligned} S_1(\boldsymbol{x}) &= \pi^{-1} \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \mid \boldsymbol{k} \mid^{-2} \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}, \\ S_2(\boldsymbol{x}) &= -(4\pi^3)^{-1} \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \mid \boldsymbol{k} \mid^{-4} \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}. \end{aligned}$$

In order to apply this fundamental solution to problem (2.4), it is necessary to get expansions for S_1 and S_2 for $|\boldsymbol{x}| \ll L$. If we scale coordinates by L, or set $L = 1$, then

$$S_1 = \frac{1}{\mid \boldsymbol{x} \mid} - C + O(\mid \boldsymbol{x} \mid^2), \tag{3.4}$$

$$\frac{\partial^2 S_2}{\partial x_l \partial x_j} = -\frac{x_j x_l}{2 \mid \boldsymbol{x} \mid^3} + \left(\frac{1}{2\mid \boldsymbol{x} \mid} - \frac{C}{3}\right)\delta_{jl} + O(|\boldsymbol{x}|^2), \tag{3.5}$$

$$u_j = U_{0j} - \frac{1}{4\pi\mu}\left\{F_j\left(\frac{1}{2\mid \boldsymbol{x} \mid} - \frac{2C}{3}\right) + \frac{\boldsymbol{F}\cdot\boldsymbol{x}}{2\mid \boldsymbol{x} \mid^3} x_j\right\} + O(|\boldsymbol{x}|^2). \tag{3.6}$$

For a simple cubic lattice the constant C is equal to $1.7601(4\pi/3)^{1/3}$. Moreover, the boundary conditions in (2.4) are to lowest order in a small-volume-fraction expansion replaced by

$$\langle \boldsymbol{u} \rangle = \frac{1}{4\pi a^2} \int_{|\boldsymbol{x}|=a} \boldsymbol{u} \, \mathrm{d}s = 0, \tag{3.7}$$

as noted by Hasimoto (1959) and carried out in detail for the analogous diffusion problem in Sangani & Acrivos (1983). Substituting (3.6) in (3.7) gives

$$\boldsymbol{U}_0 = (4\pi\mu)^{-1} \boldsymbol{F} \left(\frac{2}{3a} - \frac{2C}{3} \right) \tag{3.8}$$

and this determines $\boldsymbol{F}$ as in (1.4).

4. Inner expansion

We will use the method of matched asymptotic expansions to solve (2.5). We start with an inner expansion of the form

$$\boldsymbol{u} = \boldsymbol{u}_0 + Re\boldsymbol{u}_1 + o(Re), \tag{4.1}$$

$$p = p_0 + Rep_1 + o(Re). \tag{4.2}$$

Inserting these into (2.5) yields to $O(Re^0)$

$$\left. \begin{array}{ll} \nabla^2 \boldsymbol{u}_0 - \nabla p_0 = 0, & \\ \nabla \cdot \boldsymbol{u}_0 = 0 & \text{for } | \boldsymbol{x} - \boldsymbol{x}_n | > 1, \quad \forall \boldsymbol{n}, \\ \boldsymbol{u}_0 = 0 & \text{for } | \boldsymbol{x} - \boldsymbol{x}_n | = 1, \quad \forall \boldsymbol{n}; \end{array} \right\} \tag{4.3}$$

and to $O(Re^1)$

$$\left. \begin{array}{ll} \nabla^2 \boldsymbol{u}_1 - \nabla p_1 = (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0, & \\ \nabla \cdot \boldsymbol{u}_1 = 0 & \text{for } | \boldsymbol{x} - \boldsymbol{x}_n | > 1, \quad \forall \boldsymbol{n}, \\ \boldsymbol{u}_1 = 0 & \text{for } | \boldsymbol{x} - \boldsymbol{x}_n | = 1, \quad \forall \boldsymbol{n}. \end{array} \right\} \tag{4.4}$$

The conditions imposed on $(\boldsymbol{u}_0, p_0), (\boldsymbol{u}_1, p_1)$ do not determine them uniquely. Additional conditions are provided by matching them to the outer expansion. Specifically, we know that $\boldsymbol{u}_0$ must agree with the leading term of the outer expansion for $|\boldsymbol{x}|$ large. Since the flow is a perturbation of a uniform one, condition (2.2) in dimensionless form is

$$\boldsymbol{u}_0 \to \boldsymbol{e}_0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty.$$

Equations (4.3) with this condition are just the Stokes equations for flow past a sphere. The solution is

$$\boldsymbol{u}_0 = \boldsymbol{e}_0 - \frac{3}{4} \left(\frac{\boldsymbol{e}_0}{|\boldsymbol{x}|} + \frac{\boldsymbol{e}_0 \cdot \boldsymbol{x}}{|\boldsymbol{x}|^3} \boldsymbol{x} \right) + O(|\boldsymbol{x}|^{-3}) \tag{4.5}$$

for $|\boldsymbol{x}|$ large. This will now provide a matching condition for the first-order outer expansion and it will give the leading-term contribution to the force, which is exactly the Stokes force. The first-order outer expansion will, moreover, provide boundary conditions for the first-order inner approximation by matching, and this will determine it. We can then calculate the first-order inertial force effect in Hasimoto's formula.

5. Outer expansion

From the scaling analysis we carried out in §2, we know that inertial effects are important when $Re \sim O(a/L)$. Let us set $L = 1$ and assume that $Re = \theta a$ with θ of

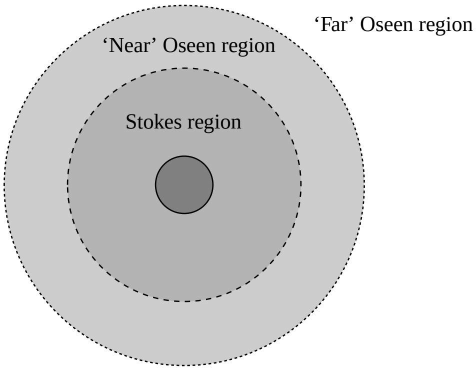

FIGURE 3. The Oseen flow region for a single particle.

order one. The outer expansion is, therefore, an expansion of solutions of (2.5) for small Re and small volume fraction.

The small-volume-fraction part of the outer expansion leads to the point force approximation, as noted in §3. We will not work this out in detail here (cf. Sangani & Acrivos 1983). We will begin instead with the point-force approximation. This means that the dimensionless equations (2.5) are to hold throughout space and the particles are replaced by a distribution of forces at the sphere centres $\boldsymbol{x}_n$:

$$\left.\begin{aligned} &\nabla^2 \boldsymbol{u} - \nabla p = Re(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \boldsymbol{F}\sum_n \delta(\boldsymbol{x}-\boldsymbol{x}_n) + \cdots, \\ &\nabla\cdot\boldsymbol{u} = 0. \end{aligned}\right\} \tag{5.1}$$

This is consistent to $O(Re)$ with the outer expansion that follows, keeping in mind that $Re = \theta a$ with θ of order one. To construct the outer expansion we need to introduce outer variables

$$\tilde{\boldsymbol{x}} = Re\boldsymbol{x}, \quad \tilde{\boldsymbol{F}} = Re\boldsymbol{F}, \quad \tilde{p} = \frac{p}{Re}.$$

Equation (5.1) is then

$$\left.\begin{aligned} &\tilde{\nabla}^2 \boldsymbol{u} - \tilde{\nabla}\tilde{p} = (\boldsymbol{u}\cdot\tilde{\nabla})\boldsymbol{u} + \tilde{\boldsymbol{F}}\sum_n \delta(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{x}}_n) + \cdots, \\ &\tilde{\nabla}\cdot\boldsymbol{u} = 0. \end{aligned}\right\} \tag{5.2}$$

where $\tilde{\boldsymbol{x}}_n$ is $\theta(n_1, n_2, n_3)$ with (n_1, n_2, n_3) integers (we have set $L = 1$).

We now turn to the form of the outer expansion. We consider first the Oseen analysis for a single particle. We have to divide the flow into several regions. One is the Stokes and the other the Oseen region, which we must divide further into the Oseen 'near' and 'far' regions, as shown in figure 3. There are two reasons for doing this. The first is that the 'far' Oseen region is different in the periodic case since the other particles will be felt. The second is that the average flow velocity of the 'near' Oseen region is different from that in the 'far' Oseen region. The average flow velocity over the full flow domain, and the 'far' Oseen region, is the uniform flow $\boldsymbol{e}_0$. But we know from the solution of the Oseen equation (Brenner & Cox 1963) that the average flow velocity over the 'near' Oseen region is $(1 + \frac{3}{8}Re)\boldsymbol{e}_0$. This, in fact, is a convenient way to define what we mean by 'near' and 'far' Oseen regions. We therefore consider the following outer expansion for the velocity:

$$\boldsymbol{u} = \boldsymbol{e}_0 + \tfrac{3}{8}Re\boldsymbol{e}_0 + Re\boldsymbol{U}_1 + o(Re). \tag{5.3}$$

We could combine the second and third terms on the right but with this definition $\boldsymbol{U}_1$ has average zero. For the pressure we consider the expansion

$$-\tilde{\nabla}\tilde{p} = (1/\theta)^3\tilde{\boldsymbol{F}} - Re\tilde{\nabla}P_1 + o(Re).$$

Substituting these expansions into the outer equation (5.2), we find that $(\boldsymbol{U}_1, P_1)$ satisfy

$$\left.\begin{aligned} &\tilde{\nabla}^2\boldsymbol{U}_1 - \tilde{\nabla}P_1 = (\boldsymbol{e}_0\cdot\tilde{\nabla})\boldsymbol{U}_1 + \boldsymbol{F}\left[\sum_{\boldsymbol{n}}\delta(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{x}}_{\boldsymbol{n}}) - (1/\theta)^3\right], \\ &\tilde{\nabla}\cdot\boldsymbol{U}_1 = 0. \end{aligned}\right\}$$

Since the leading term for the force $\boldsymbol{F}$ is $6\pi\boldsymbol{e}_0$ and additional terms do not contribute to the velocity $\boldsymbol{U}_1$ to $O(Re)$ in (5.3), we simplify further to

$$\left.\begin{aligned} &\tilde{\nabla}^2\boldsymbol{U}_1 - \tilde{\nabla}P_1 = (\boldsymbol{e}_0\cdot\tilde{\nabla})\boldsymbol{U}_1 + 6\pi\boldsymbol{e}_0\left[\sum_{\boldsymbol{n}}\delta(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{x}}_{\boldsymbol{n}}) - (1/\theta)^3\right], \\ &\tilde{\nabla}\cdot\boldsymbol{U}_1 = 0. \end{aligned}\right\} \tag{5.4}$$

This is exactly the Oseen version of equations (3.1), with zero average fields. We solve for $(\boldsymbol{U}_1, P_1)$ in the next section by generalizing Hasimoto's method.

6. The generalized periodic fundamental solution

We will deal with (5.4) in the form

$$\left.\begin{aligned} &\tilde{\nabla}^2\boldsymbol{U}_1 - \tilde{\nabla}P_1 = (\boldsymbol{e}_0\cdot\tilde{\nabla})\boldsymbol{U}_1 + \boldsymbol{F}\sum_{\boldsymbol{n}}\delta(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{x}}_{\boldsymbol{n}}), \\ &\tilde{\nabla}\cdot\boldsymbol{U}_1 = 0. \end{aligned}\right\} \tag{6.1}$$

where $\boldsymbol{F}$ now stands for $6\pi\boldsymbol{e}_0$. This is just (5.4) with the mean pressure gradient included in the new P_1.

As in Hasimoto (1959), we expand in Fourier series

$$\boldsymbol{U}_1(\tilde{\boldsymbol{x}}) = \sum_{\boldsymbol{k}}\hat{\boldsymbol{u}}_{\boldsymbol{k}}\mathrm{e}^{-2\pi\mathrm{i}\tilde{\boldsymbol{x}}\cdot\boldsymbol{k}}, \tag{6.2}$$

$$-\tilde{\nabla}P_1 = \sum_{\boldsymbol{k}}\hat{\boldsymbol{p}}_{\boldsymbol{k}}\mathrm{e}^{-2\pi\mathrm{i}\tilde{\boldsymbol{x}}\cdot\boldsymbol{k}}. \tag{6.3}$$

Here

$$\boldsymbol{k} = \frac{1}{\theta}(k_1, k_2, k_3),$$

with k_i integers. In the rest of this section we will drop the tilde to simplify writing. We will pick it up again in the next section. Using the Fourier expansion of the periodic delta functions

$$\sum_{n} \delta(\boldsymbol{x} - \boldsymbol{x}_n) = \frac{1}{\theta^3} \sum_{\boldsymbol{k}} \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{x} \cdot \boldsymbol{k}}, \tag{6.4}$$

in (6.1) we have

$$-4\pi^2 |\boldsymbol{k}|^2 \hat{\boldsymbol{u}}_{\boldsymbol{k}} + \hat{\boldsymbol{p}}_{\boldsymbol{k}} - \frac{\boldsymbol{F}}{\theta^3} = -2\pi \mathrm{i} (\boldsymbol{e}_0 \cdot \boldsymbol{k}) \hat{\boldsymbol{u}}_{\boldsymbol{k}}, \tag{6.5}$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{u}}_{\boldsymbol{k}} = 0, \tag{6.6}$$

$$\boldsymbol{k} \times \hat{\boldsymbol{p}}_{\boldsymbol{k}} = 0. \tag{6.7}$$

For $\boldsymbol{k} = 0$, (6.5) gives

$$\hat{\boldsymbol{p}}_{\boldsymbol{0}} = \frac{\boldsymbol{F}}{\theta^3}, \tag{6.8}$$

which is the mean pressure gradient that balances the point forces. For $\boldsymbol{k} \neq \boldsymbol{0}$ we have

$$\boldsymbol{k} \cdot \hat{\boldsymbol{p}}_{\boldsymbol{k}} = \frac{\boldsymbol{k} \cdot \boldsymbol{F}}{\theta^3}, \tag{6.9}$$

and from (6.5) and (6.7)

$$\hat{\boldsymbol{p}}_{\boldsymbol{k}} = \frac{(\boldsymbol{k} \cdot \boldsymbol{F})\boldsymbol{k}}{\theta^3 |\boldsymbol{k}|^2}, \tag{6.10}$$

$$\hat{\boldsymbol{u}}_{\boldsymbol{k}} = \frac{\boldsymbol{F} - (\boldsymbol{k} \cdot \boldsymbol{F})\boldsymbol{k} / |\, \boldsymbol{k} \,|^2}{2\pi \mathrm{i} \theta^3 (\boldsymbol{u}_0 \cdot \boldsymbol{k}) - 4\pi^2 \theta^3 \,|\, \boldsymbol{k} \,|^2}. \tag{6.11}$$

With these Fourier coefficients we have the representations

$$-(\nabla P_1)_j = \frac{F_j}{\theta^3} - (4\pi)^{-1} F_l \frac{\partial^2 S_1}{\partial x_l \partial x_j}, \tag{6.12}$$

$$U_{1j} = e_{0j} - (4\pi\ )^{-1} \left(F_j \tilde{S}_1 - F_l \frac{\partial^2 \tilde{S}_2}{\partial x_l \partial x_j} \right), \tag{6.13}$$

where

$$S_1 = \pi^{-1} \theta^{-3} \sum_{\boldsymbol{k} \neq \boldsymbol{0}} |\, \boldsymbol{k} \,|^{-2} \, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{x} \cdot \boldsymbol{k}}, \tag{6.14}$$

$$\tilde{S}_1 = \pi^{-1} \theta^{-3} \sum_{\boldsymbol{k} \neq \boldsymbol{0}} \left[|\, \boldsymbol{k} \,|^2 - \frac{\mathrm{i} \boldsymbol{e}_0 \cdot \boldsymbol{k}}{2\pi} \right]^{-1} \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{x} \cdot \boldsymbol{k}}, \tag{6.15}$$

$$\tilde{S}_2 = -(4\pi^3)^{-1} \theta^{-3} \sum_{\boldsymbol{k} \neq \boldsymbol{0}} |\, \boldsymbol{k} \,|^{-2} \left[|\, \boldsymbol{k} \,|^2 - \frac{\mathrm{i} \boldsymbol{e}_0 \cdot \boldsymbol{k}}{2\pi} \right]^{-1} \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{x} \cdot \boldsymbol{k}}. \tag{6.16}$$

These expressions are similar to (6.12)–(6.16) for Stokes equations. The lattice sum for $\tilde{S}_2$ is absolutely convergent everywhere, but S_1 and $\tilde{S}_1$ are only weakly convergent, in the sense of distributions. To continue with Hasimoto's approach for calculating the flow past small spheres we must estimate S_1, $\tilde{S}_1$ and $\tilde{S}_2$ for small $|\, \boldsymbol{x} \,| \ll 1$. Most of the work in estimating $\tilde{S}_1$ and $\partial^2 \tilde{S}_2 / \partial x_l \partial x_j$ as $|\, \boldsymbol{x} \,| \to 0$ is in extracting their singular part, which is the key point that makes Hasimoto's method successful. As in Hasimoto (1959), we will use the Ewald summation technique to evaluate $\tilde{S}_1$ and $\partial^2\tilde{S}_2/\partial x_l \partial x_j$.

To carry out the small-$|\,\boldsymbol{x}\,|$ expansion of the lattice sums we use the identity

$$\frac{1}{(|\boldsymbol{k}|^2 - (1/2\pi)\mathrm{i}\boldsymbol{e}_0 \cdot \boldsymbol{k})^m} = \frac{\pi^m}{\Gamma(m)} \int_0^\infty \beta^{m-1} \mathrm{e}^{-\pi(|\boldsymbol{k}|^2 - (1/2\pi)\mathrm{i}\boldsymbol{e}_0\cdot\boldsymbol{k})\beta} \mathrm{d}\beta, \qquad (6.17)$$

where m is an integer. We now introduce a more general lattice sum σ_m and transform it as follows:

$$\begin{aligned}
\sigma_m(\boldsymbol{e}_0, \theta, \boldsymbol{x}) &= \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \frac{\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}}{(|\,\boldsymbol{k}\,|^2 - (1/2\pi)\mathrm{i}\boldsymbol{e}_0\cdot\boldsymbol{k})^m} \\
&= \frac{\pi^m}{\Gamma(m)} \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \int_0^\infty \beta^{m-1} \exp\left[-\pi\,|\,\boldsymbol{k}\,|^2\,\beta - 2\pi\mathrm{i}\boldsymbol{k}\cdot\left(\boldsymbol{x} - \frac{\beta\boldsymbol{e}_0}{4\pi}\right)\right] \mathrm{d}\beta \\
&= \frac{\pi^m}{\Gamma(m)} \int_0^\infty \beta^{m-1} \left[\sum_{\boldsymbol{k}} \exp\left[-\pi\,|\,\boldsymbol{k}\,|^2\,\beta - 2\pi\mathrm{i}\boldsymbol{k}\cdot\left(\boldsymbol{x} - \frac{\beta\boldsymbol{e}_0}{4\pi}\right)\right] - 1\right] \mathrm{d}\beta.
\end{aligned} \qquad (6.18)$$

Note that σ_m implicitly depends on θ through $\boldsymbol{k}$. We will also use a slight variant of the Poisson summation formula (Courant & Hilbert 1953)

$$\sum_{\boldsymbol{k}} \exp\left[-\pi\,|\,\boldsymbol{k}\,|^2\,\beta - 2\pi\mathrm{i}\boldsymbol{k}\cdot\left(\boldsymbol{x} - \frac{\beta\boldsymbol{e}_0}{4\pi}\right)\right] = \theta^3\beta^{-3/2} \sum_{\boldsymbol{n}} \exp\left[-\frac{\pi\,|\,\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{n}} - \beta\boldsymbol{e}_0/4\pi\,|^2}{\beta}\right], \qquad (6.19)$$

in three-dimensional space. We now split the integral in (6.18) into two parts. One is from 0 to α and we use (6.19) on it, and the other is from α to ∞, where α is a suitably chosen constant. After some changes of variables we get

$$\begin{aligned}
\sigma_m(\boldsymbol{e}_0, \theta, \boldsymbol{x}) = \frac{\pi^m\alpha^m}{\Gamma(m)} \Bigg[& \alpha^{-3/2}\theta^3 \int_1^\infty \xi^{-m+1/2} \exp\left[-\pi\,|\,\boldsymbol{x} - \frac{\alpha}{4\pi\xi}\boldsymbol{e}_0\,|^2\,\frac{\xi}{\alpha}\right] \mathrm{d}\xi \\
&+ \theta^3\alpha^{-3/2} \sum_{\boldsymbol{n}\neq 0} \int_1^\infty \xi^{-m+1/2} \exp\left[-\pi\,|\,\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{n}} - \frac{\alpha}{4\pi\xi}\boldsymbol{e}_0\,|^2\,\frac{\xi}{\alpha}\right] \mathrm{d}\xi - \frac{1}{m} \\
&+ \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}} \int_1^\infty \xi^{m-1} \exp\left[-\pi\alpha\,|\,\boldsymbol{k}\,|^2\,\xi + \tfrac{1}{2}\alpha\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{e}_0)\xi\right] \mathrm{d}\xi \Bigg].
\end{aligned} \qquad (6.20)$$

We can now estimate $\tilde{S}_1$ and $\partial^2\tilde{S}_2/\partial x_l \partial x_j$ as $|\,\boldsymbol{x}\,| \to 0$. This is much more complicated than in Hasimoto's case. We give the main results here and carry out the detailed calculations in Appendix A:

$$\tilde{S}_1 = \frac{1}{|\,\boldsymbol{x}\,|} + \frac{\boldsymbol{x}\cdot\boldsymbol{e}_0}{2\,|\,\boldsymbol{x}\,|} - C(\theta, \boldsymbol{e}_0) + O(|\,\boldsymbol{x}\,|), \qquad (6.21)$$

$$\begin{aligned}
\frac{\partial^2\tilde{S}_2}{\partial x_l \partial x_j} &= -\frac{x_j x_l}{2\,|\,\boldsymbol{x}\,|^3} + \frac{1 + \frac{1}{4}\boldsymbol{x}\cdot\boldsymbol{e}_0}{2\,|\,\boldsymbol{x}\,|}\delta_{jl} \\
&\quad -\frac{1}{4}\left(\frac{x_j x_l(\boldsymbol{x}\cdot\boldsymbol{e}_0)}{2\,|\,\boldsymbol{x}\,|^3} - \frac{x_j e_{0l} + x_l e_{0j}}{2\,|\,\boldsymbol{x}\,|}\right) + C_{lj}(\theta, \boldsymbol{e}_0) + O(|\,\boldsymbol{x}\,|).
\end{aligned} \qquad (6.22)$$

Here

$$C = \frac{2}{\alpha^{1/2}} \mathrm{e}^{-\alpha/16\pi} + \frac{\alpha}{\theta^3} - \frac{1}{\alpha^{1/2}} \sum_{\boldsymbol{n} \neq 0} \int_1^\infty \xi^{-1/2} \exp\left[-\pi \mid \boldsymbol{x_n} + \frac{\alpha}{4\pi\xi} \boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right] \mathrm{d}\xi$$
$$- \frac{\alpha}{\theta^3} \sum_{\boldsymbol{k} \neq 0} \int_1^\infty \exp\left[-\pi\alpha \mid \boldsymbol{k} \mid^2 \xi + \tfrac{1}{2}\alpha \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{e}_0)\xi\right] \mathrm{d}\xi, \tag{6.23}$$

$$C_{jj} = -\frac{1}{\alpha^{1/2}}(1 + \mathrm{e}^{-\alpha/16\pi}) - \frac{\alpha^{1/2}}{32\pi} \mathrm{e}_{0j}^2 \int_1^\infty \xi^{-3/2} \mathrm{e}^{-\alpha/16\pi\xi} \mathrm{d}\xi$$
$$- \frac{\pi}{2\alpha\alpha^{1/2}} \sum_{\boldsymbol{n} \neq 0} \int_1^\infty \xi^{1/2} \left[x_{\boldsymbol{n}j}^2 \mathrm{e}^{-\pi|\boldsymbol{x_n}|^2\xi/\alpha} + (x_{\boldsymbol{n}j} + \frac{\alpha}{4\pi\xi} e_{0j})^2 \exp\left[-\pi \mid \boldsymbol{x_n} + \frac{\alpha}{4\pi\xi} \boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right]\right] \mathrm{d}\xi$$
$$+ \frac{1}{4\alpha^{1/2}} \sum_{\boldsymbol{n} \neq 0} \int_1^\infty \xi^{1/2} \left[\mathrm{e}^{-\pi|\boldsymbol{x_n}|^2\xi/\alpha} + \exp\left[-\pi \mid \boldsymbol{x_n} + \frac{\alpha}{4\pi\xi} \boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right]\right] \mathrm{d}\xi$$
$$+ \frac{\pi\alpha^2}{2\theta^3} \sum_{\boldsymbol{k} \neq 0} k_j^2 \int_1^\infty \xi \mathrm{e}^{-\pi\alpha|\boldsymbol{k}|^2\xi} (1 + \mathrm{e}^{\alpha \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{e}_0)\xi/2}) \mathrm{d}\xi$$
$$+ \frac{1}{8\pi^3\theta^3} \sum_{\boldsymbol{k} \neq 0} \frac{k_j^2 (\boldsymbol{k} \cdot \boldsymbol{e}_0)^2}{\mid \boldsymbol{k} \mid^4 (\mid \boldsymbol{k} \mid^2 - (1/2\pi) \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{e}_0)^2} \quad (j = 1, 2, 3), \tag{6.24}$$

$$C_{lj} = -\frac{\alpha^{1/2}}{32\pi} e_{0l} e_{0j} \int_1^\infty \xi^{-3/2} \mathrm{e}^{-\alpha/16\pi\xi} \mathrm{d}\xi$$
$$- \frac{\pi}{2\alpha\alpha^{1/2}} \sum_{\boldsymbol{n} \neq 0} \int_1^\infty \xi^{1/2} \left(x_{\boldsymbol{n}j} + \frac{\alpha}{4\pi\xi} e_{0j}\right)\left(x_{\boldsymbol{n}j} + \frac{\alpha}{4\pi\xi} e_{0l}\right) \exp\left[-\pi \mid \boldsymbol{x_n} + \frac{\alpha}{4\pi\xi} \boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right] \mathrm{d}\xi$$
$$+ \frac{\pi\alpha^2}{2\theta^3} \sum_{\boldsymbol{k} \neq 0} k_j k_l \int_1^\infty \xi \exp\left[-\pi\alpha \mid \boldsymbol{k} \mid^2 \xi + \frac{1}{2}\alpha \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{e}_0)\xi\right] \mathrm{d}\xi$$
$$+ \frac{1}{8\pi^3\theta^3} \sum_{\boldsymbol{k} \neq 0} \frac{k_j k_l (\boldsymbol{k} \cdot \boldsymbol{e}_0)^2}{\mid \boldsymbol{k} \mid^4 (\mid \boldsymbol{k} \mid^2 - (1/2\pi) \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{e}_0)^2} \quad (j \neq l). \tag{6.25}$$

We note that the last term in (6.24) or (6.25) is new and is needed in calculating $\partial^2 \tilde{S}_2 / \partial x_l \partial x_j$. Also, since the sum for $\tilde{S}_2$ is

$$\bar{\sigma} = \sum_{\boldsymbol{k} \neq \boldsymbol{0}} \frac{\mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{x} \cdot \boldsymbol{k}}}{\mid \boldsymbol{k} \mid^2 (\mid \boldsymbol{k} \mid^2 - (1/2\pi) \mathrm{i} \boldsymbol{e}_0 \cdot \boldsymbol{k})}$$

it is not possible to extract the singular part from it directly. However, we can write it in the form

$$\bar{\sigma} = \frac{1}{2} \left[\sum_{\boldsymbol{k} \neq \boldsymbol{0}} \left(\frac{1}{\mid \boldsymbol{k} \mid^4} + \frac{1}{(\mid \boldsymbol{k} \mid^2 - (1/2\pi) \mathrm{i} \boldsymbol{e}_0 \cdot \boldsymbol{k})^2}\right) \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \right.$$
$$\left. + \frac{1}{4\pi^2} \sum_{\boldsymbol{k} \neq \boldsymbol{0}} \frac{(\boldsymbol{e}_0 \cdot \boldsymbol{k})^2}{\mid \boldsymbol{k} \mid^4 (\mid \boldsymbol{k} \mid^2 - (1/2\pi) \mathrm{i} \boldsymbol{e}_0 \cdot \boldsymbol{k})^2} \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \right],$$

and in terms of σ_m

$$\bar{\sigma} = \tfrac{1}{2}(\sigma_2(\boldsymbol{0}, \theta, \boldsymbol{x}) + \sigma_2(\boldsymbol{e}_0, \theta, \boldsymbol{x})) + \frac{1}{8\pi^2} \sum_{\boldsymbol{k} \neq \boldsymbol{0}} \frac{(\boldsymbol{e}_0 \cdot \boldsymbol{k})^2}{\mid \boldsymbol{k} \mid^4 (\mid \boldsymbol{k} \mid^2 - (1/2\pi) \mathrm{i} \boldsymbol{e}_0 \cdot \boldsymbol{k})^2} \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}. \tag{6.26}$$

Now we can estimate $\sigma_2(\mathbf{0}, \theta, \boldsymbol{x})$ and $\sigma_2(\boldsymbol{e}_0, \theta, \boldsymbol{x})$ together with their second-order derivatives by the formulas above. Moreover, the last term and its second derivatives are all absolutely convergent.

From (6.21) and (6.22), we get the expansion for $\boldsymbol{U}_1$ as $|\boldsymbol{x}| \to 0$:

$$U_{1j} = e_{0j} - (4\pi)^{-1} \left\{ \frac{F_j}{2 \mid \boldsymbol{x} \mid}(1 + \tfrac{3}{4}\boldsymbol{x} \cdot \boldsymbol{e}_0) - F_j C + \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{2 \mid \boldsymbol{x} \mid^3} x_j - F_l C_{lj} + \frac{1}{4}\left(\frac{(\boldsymbol{F} \cdot \boldsymbol{x})(\boldsymbol{x} \cdot \boldsymbol{e}_0)}{2 \mid \boldsymbol{x} \mid^3} x_j - \frac{\boldsymbol{F} \cdot \boldsymbol{e}_0}{2 \mid \boldsymbol{x} \mid} x_j - \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{2 \mid \boldsymbol{x} \mid} e_{0j} \right) \right\}. \tag{6.27}$$

7. Calculation of the force

The expression (6.27) for the principal term of the outer expansion for the velocity gives the solution of (5.4) for $|\tilde{\boldsymbol{x}}|$ small. Using the tilde variables again we have

$$\boldsymbol{U}_1 = -\frac{3}{4}\left(\frac{\boldsymbol{e}_0}{|\tilde{\boldsymbol{x}}|} + \frac{\boldsymbol{e}_0 \cdot \tilde{\boldsymbol{x}}}{|\tilde{\boldsymbol{x}}|^3}\tilde{\boldsymbol{x}} \right) + \tfrac{3}{2}(C\boldsymbol{e}_0 + \boldsymbol{e}_0 \cdot \boldsymbol{M}) - \frac{3}{16}\left(\frac{3\tilde{\boldsymbol{x}} \cdot \boldsymbol{e}_0}{|\tilde{\boldsymbol{x}}|}\boldsymbol{e}_0 + \frac{(\boldsymbol{e}_0 \cdot \tilde{\boldsymbol{x}})^2}{|\tilde{\boldsymbol{x}}|^3}\tilde{\boldsymbol{x}} - \frac{\tilde{\boldsymbol{x}}}{|\tilde{\boldsymbol{x}}|} - \frac{\boldsymbol{e}_0 \cdot \tilde{\boldsymbol{x}}}{|\tilde{\boldsymbol{x}}|}\boldsymbol{e}_0 \right) + o(1), \tag{7.1}$$

as $|\tilde{\boldsymbol{x}}| \to 0$, where $\boldsymbol{M} = (C_{lj})_{3\times 3}$ is a matrix. Changing to the inner variables we obtain for the outer expansion of $\boldsymbol{u}$:

$$\boldsymbol{u} = \boldsymbol{e}_0 - \frac{3}{4}\left(\frac{\boldsymbol{e}_0}{|\boldsymbol{x}|} + \frac{\boldsymbol{e}_0 \cdot \boldsymbol{x}}{|\boldsymbol{x}|^3}\boldsymbol{x} \right) + \tfrac{3}{2}Re\,(C\boldsymbol{e}_0 + \boldsymbol{e}_0 \cdot \boldsymbol{M}) + \tfrac{3}{8}Re\boldsymbol{e}_0 - \frac{3}{16}Re\left(\frac{3\boldsymbol{x} \cdot \boldsymbol{e}_0}{|\boldsymbol{x}|}\boldsymbol{e}_0 + \frac{(\boldsymbol{e}_0 \cdot \boldsymbol{x})^2}{|\boldsymbol{x}|^3}\boldsymbol{x} - \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \frac{\boldsymbol{e}_0 \cdot \boldsymbol{x}}{|\boldsymbol{x}|}\boldsymbol{e}_0 \right) + O(Re^2). \tag{7.2}$$

The main term, the first term on the right-hand side, has already been matched with the zero-order inner solution. By the matching principle, the remaining terms give the following condition at infinity for the first-order inner approximation $(\boldsymbol{u}_1, p_1)$:

$$\boldsymbol{u}_1 \rightsquigarrow \tfrac{3}{2}(C\boldsymbol{e}_0 + \boldsymbol{e}_0 \cdot \boldsymbol{M}) + \tfrac{3}{8}\boldsymbol{e}_0 - \frac{3}{16}\left(\frac{3\boldsymbol{x} \cdot \boldsymbol{e}_0}{|\boldsymbol{x}|}\boldsymbol{e}_0 + \frac{(\boldsymbol{e}_0 \cdot \boldsymbol{x})^2}{|\boldsymbol{x}|^3}\boldsymbol{x} - \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \frac{\boldsymbol{e}_0 \cdot \boldsymbol{x}}{|\boldsymbol{x}|}\boldsymbol{e}_0 \right), \tag{7.3}$$

as $|\boldsymbol{x}| \to \infty$. Equation (4.4) together with this condition determines $(\boldsymbol{u}_1, p_1)$ uniquely. We will not solve for $(\boldsymbol{u}_1, p_1)$ explicitly here. Instead, we will use the result of Brenner & Cox (1963) for the force on the particle due to $(\boldsymbol{u}_1, p_1)$, as determined by the above conditions: $9\pi(C\boldsymbol{e}_0 + \boldsymbol{e}_0 \cdot \boldsymbol{M}) + \frac{9}{4}\pi\boldsymbol{e}_0$. This means that we do not need an improved version of formula (3.7), which is a considerable simplification. We note that the terms inside the second parentheses of (7.3) are odd in $\boldsymbol{x}$ and thus do not contribute to the force. Adding this force to the zero-order dimensionless force we get the drag:

$$\boldsymbol{F} = 6\pi\boldsymbol{e}_0\left(\boldsymbol{I} + \tfrac{3}{8}Re\boldsymbol{I} + \tfrac{3}{2}Re(C\boldsymbol{I} + \boldsymbol{M})\right) + \ldots \tag{7.4}$$

where C and $\boldsymbol{M} = (C_{ij})_{3\times 3}$ are given by (6.23)–(6.25) and $\boldsymbol{I}$ is the identity matrix.

This is the main result of our paper. We see that $\boldsymbol{F}$ and $\boldsymbol{e}_0$ are related in a nonlinear way now, as expected. Furthermore, the drag depends on the volume fraction (through a) and on the Reynolds number. It is thus much more complicated than both Hasimoto's (1.4) and Oseen's (1.3) formulas. In Appendix B we show that our formula for the drag reduces to the Hasimoto and Oseen formulas, in the appropriate limit.

$\theta = Re/a$	Formula (8.1)	Formula (8.2)
0.05	57.46	56.75
0.1	28.91	28.37
0.2	14.63	14.19
0.3	9.869	9.458
0.6	5.090	4.729
1.0	3.159	2.837
2.0	1.690	1.419
3.0	1.199	0.946
5.0	0.807	0.567
10.0	0.519	0.284
15.0	0.428	0.189
100.0	0.396	0.028

TABLE 1. Typical results for ΔF from our formula (8.1) and from Hasimoto's formula (8.2)

8. Numerical calculations

We now calculate numerically the coefficients C and C_{ij}, given by (6.23)–(6.25), and then the drag (7.4), for various values of the volume fraction and the Reynolds number. To simplify the computations we choose $\boldsymbol{e}_0 = (1, 0, 0)$. This means that we do not discuss directional properties of the force, which are, however, important in distinguishing Stokes from Oseen flows.

The numerical results are stated in terms of the differential drag coefficient $\Delta\boldsymbol{F}$, defined by

$$\boldsymbol{F} = 6\pi\boldsymbol{e}_0(\boldsymbol{I} + \Delta\boldsymbol{F}\,Re).$$

This is usually a 3×3 matrix, but with our choice of $\boldsymbol{e}_0$ we only need $(\Delta\boldsymbol{F})_{11}$, so we keep only the one-one component

$$\boldsymbol{F} = 6\pi\boldsymbol{e}_0(1 + \Delta F\,Re).$$

From (7.4) we have

$$\Delta F = \tfrac{3}{8} + \tfrac{3}{2}(C + C_{11}), \tag{8.1}$$

and C, C_{11} are given by (6.23) and (6.24). Thus, ΔF depends on $\boldsymbol{e}_0$ explicitly and on Re and a implicitly through their ratio $\theta = Re/a$.

The results of our numerical calculations for ΔF in (8.1) are summarized in table 1. For comparison, we also list the corresponding data from Hasimoto's formula as θ varies:

$$\Delta F = 1.7601\left(\frac{4}{3}\pi\right)^{1/3}\theta^{-1}. \tag{8.2}$$

For Oseen's formula the differential drag coefficient is equal to $\frac{3}{8}$ for all θ.

Let us examine our results for three different cases, depending on the magnitude of θ.

Case 1: θ small

In this case our result should tend to the Hasimoto's formula (1.4). This is shown in figure 4(*a*). The analytical behaviour of ΔF for θ small is discussed in Appendix B.

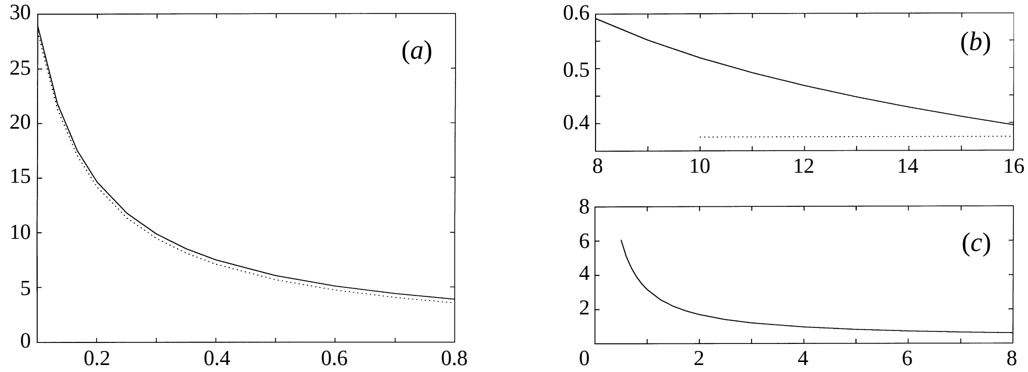

FIGURE 4. ΔF versus θ for (*a*) θ small, (*b*) θ large, (*c*) intermediate values of θ. The solid line is for ΔF computed from our formula and the dotted line in (*a*) and (*b*) is from Hasimoto's formula.

Case 2: θ large

We expect that our results tend to Oseen's formula in this case. This is shown in figure 4(*b*) where the dotted line is Oseen's formula. We do not show ΔF for θ very large because the numerical calculation of the lattice sums converges very slowly. But the agreement observed in figure 4(*b*) is good.

Case 3: θ intermediate

Our result is shown in figure 4(*c*). We see that when the Reynolds number Re and the dimensionless particle radius a, or $c^{1/3}$ with c the volume fraction, are comparable in magnitude and small, the change in the drag $\boldsymbol{F}$ depends on the ratio $\theta = Re/a$ in a complicated way. As explained in the introduction, the inertial and particle interaction effects are not simply additive. We note also that $\Delta \boldsymbol{F}$ is a full matrix for most $\boldsymbol{e}_0$ because there is directional sensitivity when inertial effects are included. This is to be expected since we approximate the flow around each particle by an Oseen flow which is not fully symmetric.

It is well known that in expressions generated by the Ewald summation technique, such as (6.23) and (6.24), the parameter α does not affect the value of C and C_{jj}. This has been verified in several cases in our numerical calculations. However, different choices of α affect the numerical convergence of C and C_{jj} greatly. The optimal α for fast convergence is hard to determine in advance. Our analysis of the limiting behaviour of C and C_{jj} in Appendix B suggests that when θ is small, α should be comparable to θ^2 and when θ is large, α should be comparable to θ. In between, we simply take $(\theta^2 + \theta)/2$.

As noted in the introduction, our results are in agreement with the phenomenological analysis of Kaneda (1986). The main qualitative difference is that the inertia-to-particle interaction parameter is $\theta = Re/a$ (or $Re/c^{1/3}$) in this paper and is $S = c/Re^2$ (or $Re/c^{1/2}$) for fixed random arrays in Kaneda (1986). This characteristic difference occurs, however, even without inertial effects. For Stokes flow (Saffman 1973), the relation between average force and average velocity for a fluid–particle system is a different function of concentration for different types of suspensions. In terms of $\Delta U = (U_0 - U)/U_0$, it is known that:

(i) for fixed periodic arrays (Hasimoto's case for a simple cubic lattice)

$$\Delta U \sim 1.7601 c^{1/3},$$

(ii) for fixed random arrays (Brinkman's case)

$$\Delta U \sim \frac{3}{\sqrt{2}} c^{1/2},$$

(iii) for free random arrays (Batchelor's sedimentation case)

$$\Delta U \sim 6.55c.$$

These relations show that fluid–particle interaction is stronger for fixed periodic arrays than for fixed random arrays, which is in turn stronger than that for random suspensions, at small volume fraction. We may expect, therefore, that inertial corrections to Batchelor's formula for sedimentation should depend on the parameter $\theta = Re/c$. As far as we know, this problem has not been analysed.

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Appendix A. Asymptotics for generalized lattice sums

Asymptotics for $\tilde{S}_1$

From its definition (6.5) we have

$$\begin{aligned}\tilde{S}_1 &= \pi^{-1}\theta^{-3}\sigma_1\\ &= \frac{1}{\alpha^{1/2}}\int_1^\infty \xi^{-1/2}\exp\left[-\pi \mid \boldsymbol{x} - \frac{\alpha}{4\pi\xi}\boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right]\mathrm{d}\xi\\ &\quad + \frac{1}{\alpha^{1/2}}\sum_{\boldsymbol{n}\neq 0}\int_1^\infty \xi^{-1/2}\exp\left[-\pi \mid -\boldsymbol{x} + \boldsymbol{x}_{\boldsymbol{n}} + \frac{\alpha}{4\pi\xi}\boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right]\mathrm{d}\xi\\ &\quad - \frac{\alpha}{\theta^3} - \frac{\alpha}{\theta^3}\sum_{\boldsymbol{k}\neq 0}\int_1^\infty \exp\left[-\pi\alpha \mid \boldsymbol{k} \mid^2 \xi + \tfrac{1}{2}\alpha\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{e}_0)\xi\right]\mathrm{d}\xi,\end{aligned} \tag{A 1}$$

The last three terms tend to finite limits as $|\boldsymbol{x}| \to 0$. So we need to analyse the first term.

Let

$$f(\boldsymbol{e}_0, \boldsymbol{x}) = \frac{1}{\alpha^{1/2}}\int_1^\infty \xi^{-1/2}\exp\left[-\pi \mid \boldsymbol{x} - \frac{\alpha}{4\pi\xi}\boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right]\mathrm{d}\xi.$$

Then by a change of variables

$$f(\boldsymbol{e}_0, \boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|}\mathrm{e}^{\boldsymbol{x}/2\cdot\boldsymbol{e}_0} 2\int_{|\boldsymbol{x}|/\alpha^{1/2}}^\infty \exp\left[-\pi\left(s^2 + \frac{|\boldsymbol{x}|^2}{16\pi^2 s^2}\right)\right]\mathrm{d}s,$$

If we let

$$g(t) = 2\int_{t/\alpha^{1/2}}^\infty \exp\left[-\pi\left(s^2 + \frac{t^2}{16\pi^2 s^2}\right)\right]\mathrm{d}s,$$

then

$$g(t) = 2\left(\int_0^\infty - \int_0^{t/\alpha^{1/2}}\right)\exp\left[-\pi\left(s^2 + \frac{t^2}{16\pi^2 s^2}\right)\right]\mathrm{d}s = g_1(t) - g_2(t).$$

Clearly $g_1(t)$ is an even function of t and $g_2(t)$ is odd. Thus

$$g(0) = g_1(0) = 2\int_0^\infty e^{-\pi s^2} ds = 1,$$
$$g'(0) = -g_2'(0) = -\lim_{t\to 0} \frac{g_2(t)}{t} = -\frac{2}{\alpha^{1/2}} e^{-\alpha/16\pi},$$
$$g''(0) = g_1''(0) \neq 0.$$

Expanding in a Taylor series we have

$$g(t) = 1 - \frac{2}{\alpha^{1/2}} e^{-\alpha/16\pi} t + O(t^2).$$

Collecting all these estimates we find that as $|\boldsymbol{x}| \to 0$

$$\begin{aligned} f(\boldsymbol{e}_0, \boldsymbol{x}) &= \frac{1}{|\boldsymbol{x}|}(1 + \tfrac{1}{2}\boldsymbol{x}\cdot\boldsymbol{e}_0 + O(|\boldsymbol{x}|^2))\left(1 - \frac{2}{\alpha^{1/2}} e^{-\alpha/16\pi}|\boldsymbol{x}| + O(|\boldsymbol{x}|^2)\right) \\ &= \frac{1}{|\boldsymbol{x}|} + \frac{\boldsymbol{x}\cdot\boldsymbol{e}_0}{2|\boldsymbol{x}|} - \frac{2}{\alpha^{1/2}} e^{-\alpha/16\pi} + O(|\boldsymbol{x}|). \end{aligned} \tag{A 2}$$

This together with (A 1) gives the estimate (6.21) for $\tilde{S}_1$.

Asymptotics for $\partial^2\tilde{S}_2/\partial x_l \partial x_j$

We use (6.26) so that the derivatives of $\tilde{S}_2$ are

$$\begin{aligned} \left.\frac{\partial^2 \tilde{S}_2}{\partial x_l \partial x_j}\right|_{\boldsymbol{x}=\boldsymbol{0}} &= -(4\pi^3)^{-1}\theta^{-3}\tfrac{1}{2}\lim_{|\boldsymbol{x}|\to 0} \frac{\partial^2}{\partial x_l \partial x_j}(\sigma_2(\boldsymbol{0},\theta,\boldsymbol{x}) + \sigma_2(\boldsymbol{e}_0,\theta,\boldsymbol{x})) \\ &\quad + \frac{1}{8\pi^3\theta^3}\sum_{\boldsymbol{k}\neq 0} \frac{k_j k_l (\boldsymbol{k}\cdot\boldsymbol{e}_0)^2}{|\,\boldsymbol{k}\,|^4\,(|\,\boldsymbol{k}\,|^2 - (1/2\pi)\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{e}_0)^2}. \end{aligned} \tag{A 3}$$

The last term is convergent. To estimate $\partial^2\tilde{S}_2/\partial x_l \partial x_j$ we have to estimate $\partial^2\sigma_2(\boldsymbol{e}_0,\theta,\boldsymbol{x})/\partial x_l \partial x_j$.

There are two cases, one when $j \neq l$ and the other when $j = l$. We analyse the first case; the analysis of the second is similar. For $j \neq l$

$$\begin{aligned} \frac{\partial^2 \sigma_2(\boldsymbol{e}_0,\theta,\boldsymbol{x})}{\partial x_l \partial x_j} &= \frac{\partial^2}{\partial x_l \partial x_j}\left\{\frac{\pi^2\alpha^2}{\Gamma(2)}\left[\alpha^{-3/2}\theta^3\sum_{\boldsymbol{n}}\int_1^\infty \xi^{-3/2}\exp\left[-\pi\,|\,\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{n}}-\frac{\alpha}{4\pi\xi}\boldsymbol{e}_0\,|^2\,\frac{\xi}{\alpha}\right]\mathrm{d}\xi\right.\right. \\ &\quad \left.\left. + \sum_{\boldsymbol{k}\neq\boldsymbol{0}} e^{-2\pi\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}\int_1^\infty \xi\exp\left[-\pi\alpha\,|\,\boldsymbol{k}\,|^2\,\xi + \tfrac{1}{2}\alpha\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{e}_0)\xi\right]\mathrm{d}\xi\right]\right\} \\ &= 4\pi^4\left[\alpha^{-3/2}\theta^3\sum_{\boldsymbol{n}}\int_1^\infty \xi^{1/2}\left(x_j - x_{\boldsymbol{n}j} - \frac{\alpha}{4\pi\xi}e_{0j}\right)\left(x_l - x_{\boldsymbol{n}j} - \frac{\alpha}{4\pi\xi}e_{0l}\right)\right. \\ &\quad \times\exp\left[-\pi\,|\,\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{n}}-\frac{\alpha}{4\pi\xi}\boldsymbol{e}_0\,|^2\,\frac{\xi}{\alpha}\right]\mathrm{d}\xi \\ &\quad \left. -\alpha^2\sum_{\boldsymbol{k}\neq 0} k_j k_l e^{-2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\int_1^\infty \xi\exp\left[-\pi\alpha\,|\,\boldsymbol{k}\,|^2\,\xi + \tfrac{1}{2}\alpha\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{e}_0)\xi\right]\mathrm{d}\xi\right]. \end{aligned} \tag{A 4}$$

Letting $|\boldsymbol{x}| \to 0$ we get

$$\lim_{|\boldsymbol{x}|\to 0} \frac{\partial^2 \sigma_2(\boldsymbol{e}_0, \theta, \boldsymbol{x})}{\partial x_l \partial x_j} = 4\pi^4 \alpha^{-3/2}\theta^3 \left(\lim_{|\boldsymbol{x}|\to 0} x_j x_l \int_1^\infty \xi^{1/2} \exp\left[-\pi \mid \boldsymbol{x} - \frac{\alpha}{4\pi\xi}\boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right] \mathrm{d}\xi \right.$$
$$\left. - \frac{\alpha}{4\pi} \lim_{|\boldsymbol{x}|\to 0} (x_j e_{0l} + x_l e_{0j}) \int_1^\infty \xi^{-1/2} \exp\left[-\pi \mid \boldsymbol{x} - \frac{\alpha}{4\pi\xi}\boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right] \mathrm{d}\xi \right) + C'_{lj} \qquad \text{(A 5)}$$

where C'_{lj} is a constant related to the first three terms of (6.25). To estimate the rest of the above expression let

$$\varphi(\boldsymbol{e}_0, \boldsymbol{x}) = \alpha^{-3/2} \int_1^\infty \xi^{1/2} \exp\left[-\pi \mid \boldsymbol{x} - \frac{\alpha}{4\pi\xi}\boldsymbol{e}_0 \mid^2 \frac{\xi}{\alpha}\right] \mathrm{d}\xi.$$

After some transformations similar to the ones for $f(\boldsymbol{e}_0, \boldsymbol{x})$, we have

$$\varphi(\boldsymbol{e}_0, \boldsymbol{x}) = \frac{\mathrm{e}^{\boldsymbol{x}/2\cdot\boldsymbol{e}_0}}{2\pi|\boldsymbol{x}|^3} \frac{2}{\pi^{1/2}} \int_{\pi|\boldsymbol{x}|^2/\alpha}^\infty t^{1/2} \exp\left[-t - \frac{|\boldsymbol{x}|^2}{16\pi t}\right] \mathrm{d}t.$$

We note that

$$\psi(s) = \frac{2}{\pi^{1/2}} \int_{s^2}^\infty t^{1/2} \exp\left[-t - \frac{s^2}{16\pi t}\right] \mathrm{d}t$$

is an even function of s and $\psi(0) = 1$. Thus

$$\psi(s) = 1 + O(s^2),$$

and hence

$$\varphi(\boldsymbol{e}_0, \boldsymbol{x}) = \frac{1}{2\pi|\boldsymbol{x}|^3}(1 + \tfrac{1}{2}\boldsymbol{x}\cdot\boldsymbol{e}_0 + O(|\boldsymbol{x}|^2)).$$

From this last estimate and (A 2) we arrive at

$$\frac{\partial^2 \sigma_2(\boldsymbol{e}_0, \theta, \boldsymbol{x})}{\partial x_l \partial x_j} = 4\pi^3\theta^3 \left(\frac{x_j x_l}{2|\boldsymbol{x}|^3} + \frac{x_j x_l (\boldsymbol{x}\cdot\boldsymbol{e}_0)}{4|\boldsymbol{x}|^3} \right)$$
$$-\pi^3 \frac{x_j e_{ol} + x_l e_{0j}}{|\boldsymbol{x}|} + C'_{lj} + O(|\boldsymbol{x}|), \qquad \text{(A 6)}$$

where $j \neq l$. This expression and the analogous one for $j = l$ give us (6.22).

Appendix B. Consistency calculations

Case 1. First we analyse (7.4) when $\theta = Re/a \to 0$. This means that inertial effects are negligible so our result should tend to the Hasimoto's formula (1.4), and this is what we show here.

In terms of a, $\boldsymbol{F}$ is

$$\boldsymbol{F} = 6\pi\boldsymbol{e}_0 \left(\boldsymbol{I} + a\cdot\tfrac{3}{8}\theta\boldsymbol{I} + a\cdot\tfrac{3}{2}\theta(C\boldsymbol{I} + \boldsymbol{M})\right) + \dots .$$

To show that this tends to Hasimoto's formula as $\theta \to 0$, we note that the Oseen term $\frac{3}{8}\theta\boldsymbol{I}$ disappears and the other term is handled by the following Lemma.

Lemma 1. *As* $\theta \to 0$, $\theta C \to 1.7601(4\pi/3)^{1/3}$ *and* $\theta\boldsymbol{M} \to -\frac{1}{3} \times 1.7601(4\pi/3)^{1/3}\boldsymbol{I}$

Proof. We give the details only for the first part since the other is similar. From

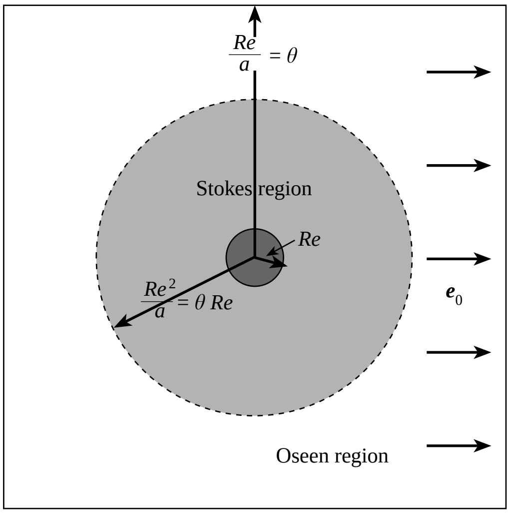

FIGURE 5. The flow domain in a period cell in terms of outer variables

(6.23), we have

$$\theta C = \frac{2\theta}{\alpha^{1/2}} e^{-\alpha/16\pi} + \frac{\alpha}{\theta^2} - \frac{\theta}{\alpha^{1/2}} \sum_{l \neq 0} \int_1^\infty \xi^{-1/2} \exp\left[-\pi \mid \boldsymbol{l} + \frac{\alpha}{4\pi\xi\theta} \boldsymbol{e}_0 \mid^2 \frac{\xi\theta^2}{\alpha}\right] d\xi$$
$$- \frac{\alpha}{\theta^2} \sum_{l \neq 0} \int_1^\infty \exp\left[-\pi \frac{\alpha}{\theta^2} |\boldsymbol{l}|^2 \xi + \frac{1}{2} \frac{\alpha}{\theta} i(\boldsymbol{l} \cdot \boldsymbol{e}_0)\xi\right] d\xi,$$

where $\boldsymbol{l}$ is in the integer lattice now. Note that α is a constant which we can change without affecting this. Thus, we may take $\alpha = \theta^2$ and then

$$\theta C = 2e^{-\theta^2/16\pi} + 1 - \sum_{l \neq 0} \int_1^\infty \xi^{-1/2} \exp\left[-\pi \mid \boldsymbol{l} + \frac{\theta}{4\pi\xi} \boldsymbol{e}_0 \mid^2 \xi\right] d\xi$$
$$- \sum_{l \neq 0} \int_1^\infty \exp\left[-\pi |\boldsymbol{l}|^2 \xi + \tfrac{1}{2}\theta i(\boldsymbol{l} \cdot \boldsymbol{e}_0)\xi\right] d\xi.$$

After some manipulation this becomes

$$\theta C = 3 - \sum_{l \neq 0} \int_1^\infty \xi^{-1/2} e^{-\pi |\boldsymbol{l}|^2 \xi} d\xi - \sum_{l \neq 0} \int_1^\infty e^{-\pi |\boldsymbol{l}|^2 \xi} d\xi,$$

as $\theta \to 0$ and the right-hand side is approximately $1.7601(4\pi/3)$, as shown by Hasimoto (1959). □

Case 2. We analyse (7.4) when $\theta \to \infty$. Since we require Re to be small, this means that the particle radius a is infinitesimal and so particle interaction is negligible. Our result (7.4) should reduce to Oseen's formula (1.3), and this is what we show now.

LEMMA 2. *As* $\theta \to \infty$, $C \to 0$ *and* $\boldsymbol{M} \to \boldsymbol{0}$.

Before proving this lemma we will look again at the the way we analysed the outer problem in §5. When we apply Hasimoto's method to the Oseen region (the outer region), we consider the whole Stokes region around each particle as an 'effective' particle (figure 5). It is clear from the figure that the radius of this 'effective' particle is the Oseen distance θRe. In our analysis, we take θ to be of order one, so the dimensionless radius is small as long as $Re \ll 1$. But when we let $\theta \to +\infty$, the radius a may also be large. This must be controlled so as not to invalidate the approximations in Hasimoto's method which require that the radius be small. So we let $\theta \to +\infty$ to recover the Oseen's case from our results, but we keep the 'effective' particle size small, i.e. $\theta Re \ll 1$.

We take these remarks into account by choosing new outer variables

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{\theta}, \quad \tilde{\boldsymbol{F}} = \frac{\boldsymbol{F}}{\theta}, \quad \tilde{p} = \theta p.$$

Equation (6.1) takes the form

$$\left.\begin{aligned} &\tilde{\nabla}^2 \boldsymbol{U}_1 - \tilde{\nabla} P_1 = \theta Re(\boldsymbol{e}_0 \cdot \tilde{\nabla})\boldsymbol{U}_1 + \boldsymbol{F}\sum_n \delta(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{x}}_n), \\ &\tilde{\nabla} \cdot \boldsymbol{U}_1 = 0. \end{aligned}\right\} \tag{B 1}$$

We now repeat the previous analysis for this equation. We get exactly the same result (7.2), except that in formula (6.23)–(6.25) for C and $\boldsymbol{M}$, θ is replaced by $1/Re$, and $\boldsymbol{e}_0$ is replaced by $\theta Re\boldsymbol{e}_0$. Now θRe enters as a parameter. To keep the notation simple we denote $1/Re$ by θ again, and $\theta Re\boldsymbol{e}_0$ by $\beta\boldsymbol{e}_0/\theta$, where $\theta \gg \beta \to +\infty$. Instead of proving Lemma 2 to recover Oseen's formula, we now prove the following.

LEMMA 3. *$C \to 0$ and $\boldsymbol{M} \to \boldsymbol{0}$ as $\theta \gg \beta \to +\infty$, where in (6.23)–(6.25) for C and $\boldsymbol{M}$, $\boldsymbol{e}_0$ is substituted by $\beta\boldsymbol{e}_0/\theta$.*

Proof. First we show that $C \to 0$. We have

$$\begin{aligned} C = {}& \frac{2}{\alpha^{1/2}}\mathrm{e}^{-\alpha/16\pi} + \frac{\alpha}{\theta^3} - \frac{1}{\alpha^{1/2}}\sum_{\boldsymbol{l}\neq 0}\int_1^\infty \xi^{-1/2}\exp\left[-\pi \mid \boldsymbol{l} + \frac{\alpha\beta}{4\pi\xi\theta^2}\boldsymbol{e}_0 \mid^2 \frac{\xi\theta^2}{\alpha}\right]\mathrm{d}\xi \\ & - \frac{\alpha}{\theta^3}\sum_{\boldsymbol{l}\neq 0}\int_1^\infty \exp\left[-\pi\frac{\alpha}{\theta^2}|\boldsymbol{l}|^2\xi + \frac{1}{2}\frac{\alpha\beta}{\theta^2}\mathrm{i}(\boldsymbol{l}\cdot\boldsymbol{e}_0)\xi\right]\mathrm{d}\xi, \end{aligned} \tag{B 2}$$

where $\boldsymbol{l}$ is an integer-valued vector. We choose $\alpha = \theta$. Then

$$\begin{aligned} C = {}& \frac{2}{\theta^{1/2}}\mathrm{e}^{-\theta/16\pi} + \frac{1}{\theta^2} - \frac{1}{\theta^{1/2}}\sum_{\boldsymbol{l}\neq 0}\int_1^\infty \xi^{-1/2}\exp\left[-\pi \mid \boldsymbol{l} + \frac{\boldsymbol{e}_0\beta}{4\pi\xi\theta} \mid^2 \xi\theta\right]\mathrm{d}\xi \\ & - \frac{1}{\theta^2}\sum_{\boldsymbol{l}\neq 0}\int_1^\infty \exp\left[-\pi\frac{1}{\theta}|\boldsymbol{l}|^2\xi + \frac{\beta}{2\theta}\mathrm{i}(\boldsymbol{l}\cdot\boldsymbol{e}_0)\xi\right]\mathrm{d}\xi. \end{aligned} \tag{B 3}$$

We can now show $C \to 0$ as $\theta \to \infty$. The first two terms go to zero, clearly. We assume that $\theta > 1$ and estimate the first sum as follows:

$$\begin{aligned} \mathrm{Sum}_1 = {}& \left|\frac{1}{\theta^{1/2}}\sum_{\boldsymbol{l}\neq 0}\int_1^\infty \xi^{-1/2}\exp\left[-\pi \mid \boldsymbol{l} + \frac{\beta\boldsymbol{e}_0}{4\pi\theta\xi} \mid^2 \xi\theta\right]\mathrm{d}\xi\right| \\ & < \frac{1}{\theta^{1/2}}\sum_{\boldsymbol{l}\neq 0}\int_1^\infty \xi^{-1/2}\exp\left[-\pi \mid \boldsymbol{l} + \frac{\boldsymbol{e}_0}{4\pi\gamma\xi} \mid^2 \xi\right]\mathrm{d}\xi = \frac{K}{\theta^{1/2}} \quad (\gamma \to +\infty) \end{aligned}$$

for some constant K which does not depend on θ. So this term vanishes like $\theta^{-1/2}$ as $\theta \to \infty$.

The last term is more difficult. We begin with the estimate

$$\begin{aligned}\text{Sum}_2 &= \left| \frac{1}{\theta^2} \sum_{\boldsymbol{l} \neq 0} \int_1^{\infty} \exp\left[-\pi \frac{1}{\theta} |\boldsymbol{l}|^2 \xi + \frac{\beta}{2\theta} \mathrm{i}(\boldsymbol{l} \cdot \boldsymbol{e}_0)\xi \right] \mathrm{d}\xi \right| \\ &\leqslant \frac{1}{\theta^2} \sum_{\boldsymbol{l} \neq 0} \int_1^{\infty} \exp\left[-\pi \frac{1}{\theta} |\boldsymbol{l}|^2 \xi \right] \mathrm{d}\xi \\ &= \frac{1}{\pi\theta} \sum_{\boldsymbol{l} \neq 0} \frac{\exp\left[-\pi \frac{1}{\theta} |\boldsymbol{l}|^2 \right]}{|\boldsymbol{l}|^2}.\end{aligned}$$

Since $f(t) = e^{-(\pi/\theta)t}/t > 0$, for $t > 0$, and decreases with t, we can estimate the last sum by comparison with an integral

$$\begin{aligned}\sum_{\boldsymbol{l} \neq 0} \frac{\exp\left[-\pi \frac{1}{\theta} |\boldsymbol{l}|^2 \right]}{|\boldsymbol{l}|^2} &\leqslant \int\int\int_{\boldsymbol{R}^3} \frac{\exp\left[-\pi \frac{1}{\theta} |\boldsymbol{x}|^2 \right]}{|\boldsymbol{x}|^2} \mathrm{d}V \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{\exp\left[-\pi \frac{1}{\theta} r^2 \right]}{r^2} r^2 \sin\alpha \mathrm{d}r \mathrm{d}\alpha \mathrm{d}\beta \\ &= K\theta^{1/2},\end{aligned}$$

where K is again a constant not depending on θ. Therefore

$$\text{Sum}_2 \leqslant \frac{1}{\pi\theta} K\theta^{1/2} \to 0$$

as $\theta \to \infty$ and $C \to 0$.

To show that $\boldsymbol{M} \to \boldsymbol{0}$, it is enough to show that $C_{jj} \to 0$. We replace in (6.24) $\boldsymbol{e}_0$ by $\beta\boldsymbol{e}_0/\theta$, let $\alpha = \theta$, and remove the θ dependence of $\boldsymbol{x_n}$ and $\boldsymbol{k}$. Then

$$\begin{aligned}C_{jj} = &-\frac{1}{2\theta^{1/2}}(1 + \mathrm{e}^{-\theta/16\pi}) - \frac{\theta^{1/2}}{32\pi}\left(\frac{\beta}{\theta}\right)^2 e_{0j}^2 \int_1^{\infty} \xi^{-3/2} e^{-\theta/16\pi\xi} \mathrm{d}\xi \\ &-\frac{\pi}{2\theta\theta^{1/2}} \sum_{\boldsymbol{l} \neq 0} \int_1^{\infty} \xi^{1/2} \left[\theta^2 l_j^2 \mathrm{e}^{-\pi\theta|\boldsymbol{l}|^2\xi} + \theta^2 \left(l_j + \frac{\beta}{4\pi\theta\xi} e_{0j} \right)^2 \exp\left[-\pi|\boldsymbol{l} + \frac{\beta}{4\pi\theta\xi} \boldsymbol{e}_0|^2 \theta\xi \right] \right] \mathrm{d}\xi \\ &+\frac{1}{4\theta^{1/2}} \sum_{\boldsymbol{l} \neq 0} \int_1^{\infty} \xi^{1/2} \left[\mathrm{e}^{-\pi\theta|\boldsymbol{l}|^2\xi} + \exp\left[-\pi\theta \mid \boldsymbol{l} + \frac{\beta}{4\pi\theta\xi} \boldsymbol{e}_0 \mid^2 \xi \right] \right] \mathrm{d}\xi \\ &+\frac{\pi}{2\theta^3} \sum_{\boldsymbol{l} \neq 0} l_j^2 \int_1^{\infty} \xi \mathrm{e}^{-\pi|\boldsymbol{l}|^2/\theta\xi} \left(1 + \exp\left[\frac{\beta}{2\theta} \mathrm{i}(\boldsymbol{l} \cdot \boldsymbol{e}_0)\xi \right) \right] \mathrm{d}\xi \\ &+\frac{1}{8\pi^3\theta} \sum_{\boldsymbol{l} \neq 0} \frac{l_j^2 (\beta\boldsymbol{l} \cdot \boldsymbol{e}_0)^2}{\mid \boldsymbol{l} \mid^4 (\mid \boldsymbol{l} \mid^2 - (1/2\pi)\mathrm{i}\beta\boldsymbol{l} \cdot \boldsymbol{e}_0)^2} \quad (j = 1, 2, 3).\end{aligned}$$

Only the second and last terms need analysis. The other terms can be handled in the same way as in showing $C \to 0$.

For the second term, we note that

$$\theta^{1/2}\int_1^\infty \xi^{-3/2}\mathrm{e}^{-\theta/16\pi\xi}\mathrm{d}\xi = \int_1^\infty \left(\frac{\xi}{\theta}\right)^{-3/2}\mathrm{e}^{-\theta/16\pi\xi}\mathrm{d}\left(\frac{\xi}{\theta}\right)$$
$$= \int_{1/\theta}^\infty s^{-3/2}\mathrm{e}^{-1/16\pi s}\mathrm{d}s$$
$$\leqslant \int_0^\infty s^{-3/2}\mathrm{e}^{-1/16\pi s}\mathrm{d}s = K$$

where K is some constant not depending on θ. So this term will vanish as $\beta/\theta \to 0$ ($\beta \ll \theta$).

We consider now the last term and take $\boldsymbol{e}_0 = (1,0,0)$ for simplicity. We have, therefore

$$\mathrm{Sum}_3 = \frac{1}{8\pi^3\theta}\sum_{\boldsymbol{l}\neq 0}\frac{l_j^2(\beta l_1)^2}{|\,\boldsymbol{l}\,|^4\,(|\,\boldsymbol{l}\,|^2 - (1/2\pi)\mathrm{i}\beta l_1)^2},$$

to be estimated for $\theta \gg \beta \to +\infty$. We show that

$$\left|\frac{1}{\beta}\sum_{\boldsymbol{l}\neq 0}\frac{l_j^2(\beta l_1)^2}{|\,\boldsymbol{l}\,|^4\,(|\,\boldsymbol{l}\,|^2 - (1/2\pi)\mathrm{i}\beta l_1)^2}\right| \leqslant K \tag{B 4}$$

as $\beta \to +\infty$, for some constant K that does not depend on β.

With this estimate, it is clear that $|\mathrm{Sum}_3| = O(\beta/\theta) \to 0$ as $\theta \gg \beta \to +\infty$ and the proof of Lemma 3 is complete.

To prove (B 4), we will prove the slightly stronger estimate

$$\mathrm{Sum} = \left|\frac{1}{\beta}\sum_{l_1>0}\sum_{l_2,l_3}\frac{(\beta l_1)^2}{|\,\boldsymbol{l}\,|^2\,(|\,\boldsymbol{l}\,|^2 - \mathrm{i}\beta l_1)^2}\right| \leqslant K$$

as $\beta \to +\infty$ and again K does not depend on β. We split the sum into two parts:

$$\mathrm{Sum} \leqslant \frac{1}{\beta}\sum_{l_1>0}\sum_{|\boldsymbol{l}|^2>\beta l_1}\frac{(\beta l_1)^2}{|\,\boldsymbol{l}\,|^2\,|\,|\,\boldsymbol{l}\,|^2 - \mathrm{i}\beta l_1|^2} + \frac{1}{\beta}\sum_{l_1>0}\sum_{|\boldsymbol{l}|^2\leqslant\beta l_1}\frac{(\beta l_1)^2}{|\,\boldsymbol{l}\,|^2\,|\,|\,\boldsymbol{l}\,|^2 - \mathrm{i}\beta l_1|^2}$$
$$= (\mathrm{I}) + (\mathrm{II}).$$

Consider (II) first. We have

$$\beta l_1 - l_1^2 \geqslant l_2^2 + l_3^2 \geqslant 0,$$

so that $l_1 \leqslant \beta$. Thus

$$(\mathrm{II}) \leqslant \frac{1}{\beta}\sum_{0<l_1\leqslant\beta}\;\sum_{l_2^2+l_3^2\leqslant\beta l_1-l_1^2}\frac{1}{|\boldsymbol{l}|^2} \leqslant \frac{K}{\beta}\sum_{0<l_1\leqslant\beta}\int_0^{\beta l_1 - l_1^2}\frac{\mathrm{d}r^2}{l_1^2 + r^2}$$
$$\leqslant \frac{K}{\beta}\sum_{0<l_1\leqslant\beta}\log\frac{\beta}{l_1} \leqslant \frac{K}{\beta}\int_1^\beta \log\frac{\beta}{x}\mathrm{d}x \leqslant K\int_0^1 \log\frac{1}{t}\mathrm{d}t = K.$$

Here K stands for different constants that do not depend on β.

Consider (I). We have

$$
\begin{aligned}
(\mathrm{I}) &\leqslant \frac{1}{\beta}\sum_{l_1>0}\sum_{l_2^2+l_3^2>\beta l_1-l_1^2}\frac{(\beta l_1)^2}{|\boldsymbol{l}|^6}\\
&\leqslant \frac{1}{\beta}\sum_{l_1\geqslant\beta}\sum_{l_2,l_3}\frac{(\beta l_1)^2}{|\boldsymbol{l}|^6}+\frac{1}{\beta}\sum_{l_1<\beta}\sum_{l_2^2+l_3^2>\beta l_1-l_1^2}\frac{(\beta l_1)^2}{|\boldsymbol{l}|^6}\\
&\leqslant \frac{K}{\beta}\sum_{l_1\geqslant\beta}(\beta l_1)^2\int_0^\infty\frac{\mathrm{d}r^2}{(l_1^2+r^2)^3}+\frac{K}{\beta}\sum_{l_1<\beta}(\beta l_1)^2\int_{\beta l_1-l_1^2}^\infty\frac{\mathrm{d}r^2}{(l_1^2+r^2)^3}\\
&\leqslant \frac{K}{\beta}\sum_{l_1\geqslant\beta}(\beta l_1)^2\frac{1}{l_1^4}+\frac{K}{\beta}\sum_{l_1<\beta}(\beta l_1)^2\frac{1}{(\beta l_1)^2}\leqslant K\beta\sum_{l_1\geqslant\beta}\frac{1}{l_1^2}+K\\
&\leqslant K\beta\int_\beta^\infty\frac{\mathrm{d}x}{x^2}+K=K,
\end{aligned}
$$

and again K denotes constants independent of β. So this term is also bounded and the proof of (B 4) complete. □

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